Recall: G, H groups

$\Phi : G \to H$ is a homomorphism

if $\Phi(ab) = \Phi(a)\Phi(b)$ for all a, b in G

Question: Given 2 groups G, H,

how many homomorphisms are there, $\Phi : G \to H$.

Examples: Assume G is cyclic

have seen: in this case $G \cong \mathbb{Z}$ (if infinite)

or $G \cong \mathbb{Z}_n$ (if finite)

Crucial observation:

If G is cyclic with generating element a

i.e. $G = \langle a \rangle$ $\Rightarrow$

$\Rightarrow$ any hom. $\Phi: G \to H$
is already uniquely determined if we know $\Phi(a)$

indeed any element of G is of the form a^j

$\Rightarrow$ $\Phi(a^j) \underset{\uparrow}{=} \Phi(a)^j$ = determined by $\Phi(a)$

hom. properties

Example: Determine all homomorphisms from $G = \mathbb{Z}$
into $H \subset A_4$, $H = \{id, (12)(34), (13)(24), (14)(23)\}$

Solution: Define for any $h \in H$ the hom.

$\Phi_h : 1 \to h$

$\Rightarrow$ $\Phi_h(j) = \Phi_h(\underbrace{1 + 1 + \ldots + 1}_{j \text{ times}}) = \Phi_h(1)^j = h^j$

(use add. notation) for $\mathbb{Z}$

use multipl. not. for H

Φ_h does define a hom.

If $h = id$. $\Rightarrow$ $\Phi_h(j) = id$ for all $j \in \mathbb{Z}$

If $ord(h) = 2$ $\Rightarrow$ $\Phi_h(j) = \begin{cases} e & \text{if } j \text{ is even} \\ h & \text{if } j \text{ is odd.} \end{cases}$

$\Rightarrow$ ker of $\Phi_h = \begin{cases} \mathbb{Z} & \text{if } h = e \\ 2\mathbb{Z} & \text{if } h \neq e \end{cases}$

$\Phi_h(\mathbb{Z}) = \langle h \rangle$

<u>General Fact.</u> : There exists exactly one hom. $\Phi_h : \mathbb{Z} \to H$ for every elem. $h \in H$.
and these are all possible group hom. from $\mathbb{Z}$ to the group H

true for ANY group H.

(2) Find all hom. from $\mathbb{Z}_3$ into H, $H \subset A_4$ as before.

Solution: $\mathbb{Z}_3$ is cyclic $\Rightarrow$

any hom. Φ already completely determined by $\Phi(1)$

Question: What can we take for $\Phi(1)$?

Try $\Phi(1) = id$. $\Rightarrow$ $\Phi(j) = id^j = id$ for all $j \in \mathbb{Z}$

$\Rightarrow$ $\Phi(j+k) = id = id \circ id = \Phi(j)\Phi(k)$

✓

Try $\Phi(1) = h \neq id$

Say $h = (12)(34)$

Does this define a homomorphism?

$\Rightarrow \Phi(2) = h^2 = id$

$\boxed{\Phi(3) = h^3 = h}$

$\uparrow \mathbb{Z}_3$

$3 \bmod 3 = 0$

$\Rightarrow \boxed{\Phi(3) = \Phi(0) = id}$

$\Rightarrow$ no hom. $\Phi: \mathbb{Z}_3 \to H$ possible

with $\Phi(1) = h \neq id$

<u>Remark:</u> Nonexistence of such hom. can also be seen from the fact $\operatorname{ord} \Phi(g) \mid \operatorname{ord}(g)$ for any $g \in G$

any hom. $\Phi: G \to H$

in our example: $\operatorname{ord}(1) = 3$ in $\mathbb{Z}_3$

$\operatorname{ord}(h) = 2$ for $h \in H$, $h \neq id$.

$\Rightarrow$ no hom. Φ which would map 1 to h.

Essentially we have proved the following theorem:

Theorem: Let H be any group, $G = \langle a \rangle$ cyclic.

ⓐ If $ord(a) = \infty$ (i.e. $G \cong \mathbb{Z}$)

$\Rightarrow$ there exists a hom $\Phi_h : a \to h$ for any $h \in H$.

These are all possible hom. $G \to H$

ⓑ If $ord(a) = n$

$\Rightarrow$ $\Phi_h : a \to h$ defines a homom. if and only if $ord(h) \mid n$.

Example: Find all hom. $\Phi : \mathbb{Z}_6 \to \mathbb{Z}_9$

Sol. by theorem $\Phi(1) = j$ defines a hom.

$\Leftrightarrow$ $ord(j) \mid \underset{\uparrow \mathbb{Z}_6}{ord(1)} = 6$

Recall: $j \in \mathbb{Z}_9 \Rightarrow \operatorname{ord}(j) = \dfrac{9}{\gcd(j,9)} = \begin{cases} 1 & j=0 \\ 3 & j=3,6 \\ 9 & \text{otherwise.} \end{cases}$

$\Rightarrow \operatorname{ord}(j) \mid 6 \iff j \in \{0,3,6\}$

$\Rightarrow$ have exactly 3 homom. $\Phi: \mathbb{Z}_6 \to \mathbb{Z}_9$

- $\Phi(k) = 0$ for all $k \in \mathbb{Z}_6$ $\quad j=0$
- $\Phi(k) = 3k \bmod 9$ " " " $\quad j=3$
- $\Phi(k) = 6k \bmod 9$ " " " $\quad j=6$

Fundamental Theorem of Finite Abelian Groups

main result:
as stated in book: $G \cong$ direct product of cyclic groups

arbitrary finite abelian group

alternative statement (better for explicit calculations)

1. If $|G| = p^a \Rightarrow G \cong \mathbb{Z}_{p^{a_1}} \oplus \mathbb{Z}_{p^{a_2}} \oplus \dots \oplus \mathbb{Z}_{p^{a_r}}$

   where $a_1 + a_2 + \dots + a_r = a$

   Def. $(a_1, a_2, \dots a_r)$ is a partition of a
   if $a_1 \geq a_2 \geq \dots \geq a_r \geq 0$ integers
   and $a_1 + a_2 + \dots + a_r = a$

   $\Rightarrow$ If $\mathrm{Par}(a) = \#$ partitions of a
   then there are exactly $\mathrm{Par}(a)$ nonisom. groups G of order p^a

(2) If $|G| = p_1^{a(1)} p_2^{a(2)} \cdots p_s^{a(s)}$

$\Rightarrow G = G_{p_1} \oplus G_{p_2} \oplus \cdots \oplus G_{p_s}$

where $|G_{p_i}| = p^{a(i)}$.

and there are exactly

$Par(a(1)) \cdot Par(a(2)) \cdot \ldots \cdot Par(a(s))$ nonisom. groups G

wite $|G| = p_1^{a(1)} \cdots p_s^{a(s)}$

Example: how many nonisom. abelian groups of order 100?

Sol. $100 = 2^2 \cdot 5^2$

$G = G_2 \oplus G_5$

$|G_2| = 2^2 \longrightarrow \mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (2 possibilities)

$|G_5| = 5^2 \xrightarrow[\text{2 poss.}]{} \mathbb{Z}_{25}$ or $\mathbb{Z}_5 \oplus \mathbb{Z}_5$

$\Rightarrow$ have $Par(2) \cdot Par(2) = 2 \cdot 2 = 4$ possibilities.